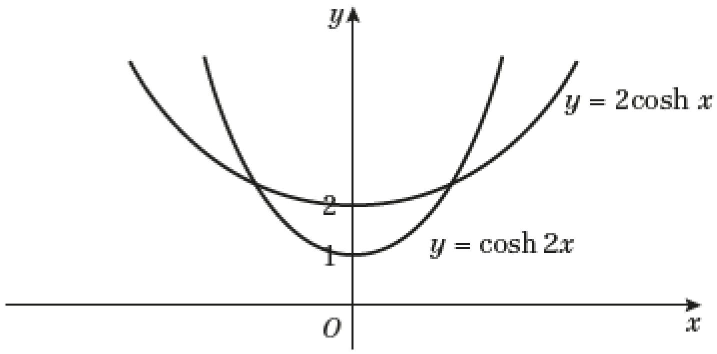
Exercise 1B

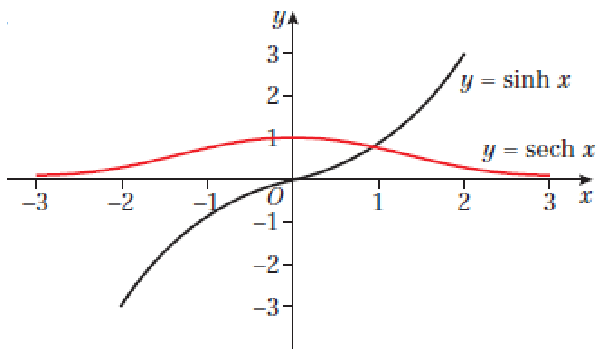
1



For $f(x) = \cosh x$,
$f(2x) = \cosh 2x$, a horizontal sketch of scale factor $\frac{1}{2}$.

For $f(x) = \cosh x$,
$2f(x) = 2\cosh x$, a vertical stretch of scale factor 2.

2 a



2 b The curves $y = \operatorname{sech} x$ and $y = \sinh x$ meet when:

$$\operatorname{sech} x = \sinh x$$

$$\frac{2}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{2}$$

$$\frac{2e^x}{e^{2x} + 1} = \frac{e^{2x} - 1}{2e^x}$$

$$4e^{2x} = \left(e^{2x} + 1\right)\left(e^{2x} - 1\right)$$

$$4e^{2x} = e^{4x} - 1$$

$$e^{4x} - 4e^{2x} - 1 = 0$$

Let $y = e^{2x}$:

$$y^2 - 4y - 1 = 0$$

$$y = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{4 \pm 2\sqrt{5}}{2}$$

$$= 2 \pm \sqrt{5}$$

Since $y = e^{2x}$:

$e^{2x} = 2 + \sqrt{5}$ or $e^{2x} = 2 - \sqrt{5}$

When $e^{2x} = 2 + \sqrt{5}$

$$2x = \ln\left(2 + \sqrt{5}\right)$$

$x = \frac{1}{2}\ln\left(2 + \sqrt{5}\right)$ as required

When $e^{2x} = 2 - \sqrt{5}$

$e^{2x} < 0$, which would be impossible so this gives no further solutions.

3 **a** $\mathrm{f}(x) \in \mathbb{R}$ (All real numbers)

Check the graph of each hyperbolic function to see which *y* values are possible.

b $\mathrm{f}(x) \geqslant 1$

c $-1 < \mathrm{f}(x) < 1$

$|\mathrm{f}(x)| < 1$

d $\mathrm{f}(x) = \operatorname{sech} x,\ x \in \mathbb{R}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

When $x = 0$, $\operatorname{sech} x = \frac{1}{1} = 1$

As $x \to \infty$, $\cosh x \to \infty$, so $\operatorname{sech} x \to 0$

As $x \to -\infty$, $\cosh x \to -\infty$, so $\operatorname{sech} x \to 0$

The *x*-axis is an asymptote to the curve.

Therefore $\mathrm{f}(x) = \operatorname{sech} x,\ x \in \mathbb{R}$ has the range:

$0 < \mathrm{f}(x) \leq 1$

e $\mathrm{f}(x) = \operatorname{cosech} x,\ x \in \mathbb{R},\ x \neq 0$

$\operatorname{cosech} x = \dfrac{1}{\sinh x}$

For positive *x*, as $x \to 0$, $\operatorname{cosech} x \to \infty$

For negative *x*, as $x \to 0$, $\operatorname{cosech} x \to -\infty$

As $x \to \infty$, $\sinh x \to \infty$, so $\operatorname{cosech} x \to 0$

As $x \to -\infty$, $\sinh x \to -\infty$, so $\operatorname{cosech} x \to 0$

The *x*-axis and *y*-axis are asymptotes to the curve.

Therefore $\mathrm{f}(x) = \operatorname{cosech} x,\ x \in \mathbb{R},\ x \neq 0$ has the range:

$\mathrm{f}(x) \in \mathbb{R},\ x \neq 0$

f $\mathrm{f}(x) = \coth x,\ x \in \mathbb{R},\ x \neq 0$

$\coth x = \dfrac{1}{\tanh x}$

For positive *x*, as $x \to 0$, $\coth x \to \infty$

For negative *x*, as $x \to 0$, $\coth x \to -\infty$

As $x \to \infty$, $\tanh x \to 1$, so $\coth x \to 1$

As $x \to -\infty$, $\tanh x \to -1$, so $\coth x \to -1$

So the *y*-axis is an asymptote to the curve as are the lines $y = -1$ and $y = 1$

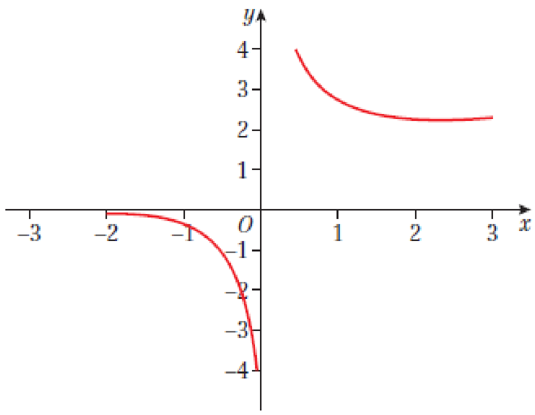
Therefore $\mathrm{f}(x) = \coth x,\ x \in \mathbb{R},\ x \neq 0$ has the range:

$\mathrm{f}(x) < -1$ or $\mathrm{f}(x) > 1$

4 a $f(x) = 1 + \coth x,\ x \in \mathbb{R},\ x \neq 0$

$\coth x = \dfrac{1}{\tanh x}$

For positive x, as $x \to 0$, $\coth x \to \infty$, so $1 + \coth x \to \infty$
For negative x, as $x \to 0$, $\coth x \to -\infty$, so $1 + \coth x \to -\infty$
As $x \to \infty$, $\tanh x \to 1$, so $\coth x \to 1$, so $1 + \coth x \to 2$
As $x \to -\infty$, $\tanh x \to -1$, so $\coth x \to -1$, so $1 + \coth x \to 0$



b The curve has asymptotes at:
$x = 0$, $y = 0$ and $y = 2$

5 a $y = 3\tanh x,\ x \in \mathbb{R},\ x \neq 0$

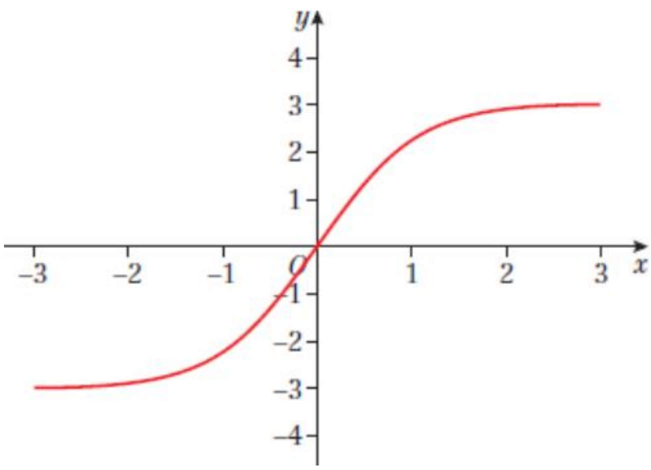
$3\tanh x = \dfrac{3\sinh x}{\cosh x}$

When $x = 0$, $3\tanh x = \dfrac{0}{1} = 0$

When x is large and positive, $3\sinh x \approx \dfrac{3}{2}e^{x}$ and $\cosh x \approx \dfrac{1}{2}e^{x}$, so $\tanh x \approx 3$

When x is large and negative, $3\sinh x \approx -\dfrac{3}{2}e^{-x}$ and $\cosh x \approx -\dfrac{1}{2}e^{-x}$, so $\tanh x \approx -3$

As $x \to \infty$, $3\tanh x \to 3$
As $x \to -\infty$, $3\tanh x \to -3$



b The curve has asymptotes at:
$y = -3$ and $y = 3$

Challenge

$$y = \sinh x + \cosh x$$
$$= \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}$$
$$= e^x$$